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COMMENT

On the problem of measuring fractal dimensions of random interfaces

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Abstract Various authors have shown that the experimentally observed fractal dimension of a random interface may depend drastically on the size of the probe used for the measurements. We propose here a general explanation for such dependence that we can explicitly calculate for a simple model.

Physical properties of external surfaces of fractal objects play an important role in many scientific domains ranging from electrochemistry, diffusion processes and chromatography to heterogeneous catalysis, hydrology and petrology. The dimension d_H of the fractal surface is in general not given by $d_f - 1$ where d_f is the dimension (fractal or Euclidean) of the object itself, as is the case in homogeneous spaces [1]. Moreover to characterise the surface completely it may be necessary to consider a hierarchy of fractal dimensions depending on the set of physical properties one wants to examine [2].

Recently, several experiments have attempted to relate measurements of particle adsorption or chemical reactions to the structure of fractal surfaces. For example, from the quantity of adsorbed particles one can determine experimentally a fractal dimension d_e associated with the surface [3, 4]. This experimental technique has been recently examined numerically by Grossman and Aharony [5] when studying the fractal dimension of the percolation hull on a square lattice. It appears that the measured fractal dimension d_e depends on the size of the adsorbed particles. For particle diameters less than or equal to the lattice constant one finds [6-8] the hull dimension $d_H = \frac{7}{4}$, while for diameters larger than the open space between next-nearest-neighbour sites the adsorbed particles are not able to enter large 'bays' which are present in the hull. Grossman and Aharony find in this case $d_e = 1.37 \pm 0.03$ in place of the expected $d_H = 1.75$, while Meakin and Family [9] obtain $d_e = 1.34$.

A similar problem might also appear when studying the fractal dimension of an object from its computerised picture. In this case the pixel size is the limiting parameter. Recently, Shaw [10] has investigated the movement of a drying front in porous 2D materials. The experimental patterns are digitised to make them accessible to a numerical study. The front is shown to have a fractal morphology, with dimension 1.38, smaller than the dimension 1.75 expected for an invasion percolation front. However, the pixel size of the front image is larger than the grain size of the porous material [11]. Then apertures in the front smaller than the pixel size cannot be detected, which certainly may account for the above discrepancy.

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To explain these effects we study here a simple model for the fractal interface, which is simple enough to be treated rigorously. The model is as follows. We start with the von Koch curve which is represented in figure 1. Only two levels of iteration are shown. Here the Koch curve will be taken as the interface between two media with different properties. For example, the hatched region is matter and the white region is empty. Using a sea analogy, we define B_1 and B_2 as 'bays' in this interface. The large bay B_1 is created at level 1 of the hierarchy, the smaller bays B_2 are created at level 2.

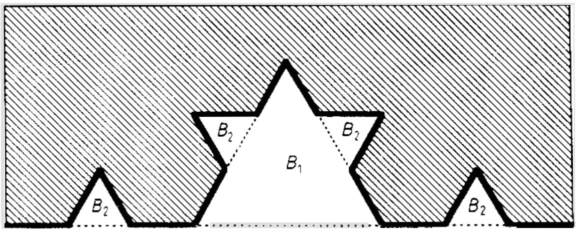

Figure 1. Two iteration steps for the modified Koch curve described in the text, with the idealised external surface of a material represented by the hatched region. The jetties with variable apertures are shown here as broken lines.

In more realistic interfaces (e.g. percolation hull), bays have apertures of variable size. To introduce this feature in the von Koch curve we add a 'jetty' leaving a variable open space of length λ_i for a boat to penetrate in the bay B_i. The jetties and the remaining apertures are represented in figure 1 by broken lines.

The problem of measuring the number of molecules adsorbed on a fractal surface is now replaced by the problem of measuring the length of coast, accessible to a boat of width b. Here b stands for the diameter of the adsorbed molecule; alternatively, it may also represent the pixel size of digitalisation. If $b > \lambda_i$, the bay is not accessible; if $b < \lambda_i$, the bay is accessible. If the apertures λ_i are distributed with a probability distribution $\bar{\omega}_i(\lambda_i)$ for all the bays at the same level of hierarchy i, each bay is accessible with a probability $p_i(b)$

$$p_i(b) = 1 - \int_0^b \bar{\omega}_i(\lambda_i)\, \mathrm{d}\lambda_i. \tag{1}$$

In order to determine the accessible part of the coast, we consider a 'seashore' segment S_{i-1} at level $i-1$, with mass M_{i-1}. At level i four new segments S_i (with mass M_i) are generated if the bay is open. Only two segments plus the jetty J_i are generated if the bay is inaccessible (see figure 2). In figure 2, a 'segment' S_i is represented by a simple line.

We consider in the following two different choices for the jetties. (a) They are considered as part of the seashore, playing exactly the same role as segments S_i in the iteration (figure 2(a)), or (b) they are not considered as part of the seashore. Their length L_i is added to the seashore length only if the bay is not accessible, and ignored otherwise (figure 2(b)). Accordingly, successive levels i are connected by the recursion relation

$$M_{i-1} \Rightarrow p_i \times 5M_i + (1-p_i) \times 3M_i \tag{2a}$$

$$M_{i-1} \Rightarrow p_i \times 4M_i + (1-p_i)(2M_i + L_i) \tag{2b}$$

where p_i (respectively $1-p_i$) is the probability for the bay to be accessible (respectively closed).

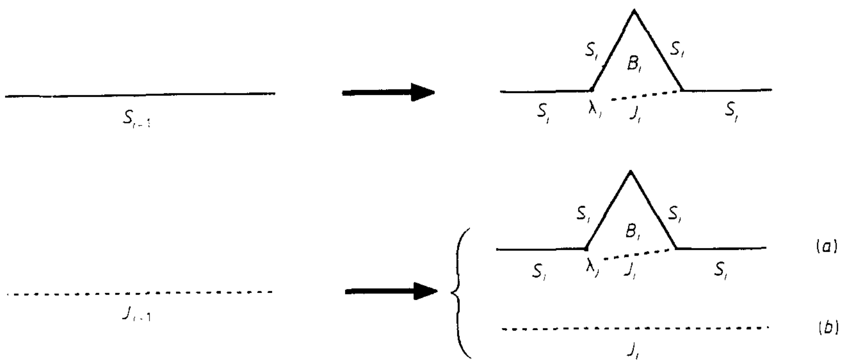

Figure 2. This figure schematises the iterating process of our simple hierarchical model. A segment S_{i-1}, symbolising a portion of Koch curve at level $i-1$, is replaced at level i by the elements shown on the right. The bay B_i is accessible with probability $p_i(b)$ or closed with probability $1-p_i(b)$ depending on the size of the aperture λ_i with respect to the probe width b. During this iteration the jetty J_{i-1} can be iterated like S_{i-1} (case (a)) or left unchanged (case (b)).

Let N be the highest level in the hierarchy where the minimum possible length a is reached and hence $M_N = a$. The total mass M after N iteration steps follows from (2), for the different cases (a) and (b):

$$M = \{(1+2p_1/3)\times\ldots\times(1+2p_N/3)\times L\} \tag{3a}$$

$$\begin{aligned} M = \{&(2/3)^N(1+p_1)\times\ldots\times(1+p_N) \\ &+(2/3)^{N-1}(1+p_1)\times\ldots\times(1+p_{N-1})[(1-p_N)/3] \\ &+(2/3)^{N-2}(1+p_1)\times\ldots\times(1+p_{N-2}) \\ &\times[(1-p_{N-1})/3]+\ldots+[(1-p_1)/3]\}\times L. \end{aligned} \tag{3b}$$

Here $L=3^N a$ is the length of the initial segment S_0.

Now consider a jetty of length L_i at the entrance of a bay B_i. This jetty leaves an open space λ_i. We suppose that the smallest possible aperture is a_0. This aperture a_0 is to be compared, e.g., with the aperture $a\sqrt{2}$ in the case of the hull on a square lattice ($a\sqrt{3}$ for the triangular lattice). At level 'i' the maximum diameter of the aperture is L_i, i.e. $a_0 \leqslant \lambda_i \leqslant L_i$. By definition, the distribution of apertures is normalised and

$$\int_{a_0}^{L_i} \bar{\omega}_i(\lambda_i)\, \mathrm{d}\lambda_i = 1. \tag{4}$$

First suppose that all the bays are completely open (as is the usual von Koch curve). Here we have simply

$$\bar{\omega}_i(\lambda_i) = \delta(\lambda_i - L_i). \tag{5}$$

We want to measure the length of the coast accessible to a boat of width b; b satisfies the inequality

$$L_{n+1} < b < L_n. \tag{6}$$

That defines a step $n+1$ of iteration above which the bays are not accessible for boats of width b. Thus we have

$$p_i = \begin{cases} 1 & i = 1, \ldots, n \\ 0 & i = n+1, \ldots N. \end{cases}$$

Then from ($3a$, b) (using here L_n as unit length) one easily recovers that

$$M = 5^n L_n \qquad L = 3^n L_n \tag{7a}$$

$$M = 4^n L_n \qquad L = 3^n L_n . \tag{7b}$$

From the definition of the fractal dimension d_f

$$M = (L_n)^{1-d_f} L^{d_f} \tag{8}$$

we obtain immediately the conventional results for cases (a) and (b), respectively

$$d_e = d_f = \ln 5/\ln 3 \tag{9a}$$

$$d_e = d_f = \ln 4/\ln 3 \tag{9b}$$

which are independent of b. We find the same result for d_e, if the bays can have apertures of different sizes, but the width of the boat is small enough to enter all bays.

Now consider the most interesting case (figure 3(a)) where the diameter of the aperture can accept discrete values a_0, a_0', a_0'', ..., L_i, with probability q, q', q'', ..., p, i.e.

$$\bar{\omega}_i(\lambda_i) = q\delta(\lambda_i - a_0) + q'\delta(\lambda_i - a_0') + \ldots + p\delta(\lambda_i - L_i) \tag{10}$$

where, due to normalisation, $p = 1 - (q + q' + q'' + \ldots)$, but b is now *greater* than a_0. In this case the length of coast reached by a boat of width b does depend on b.

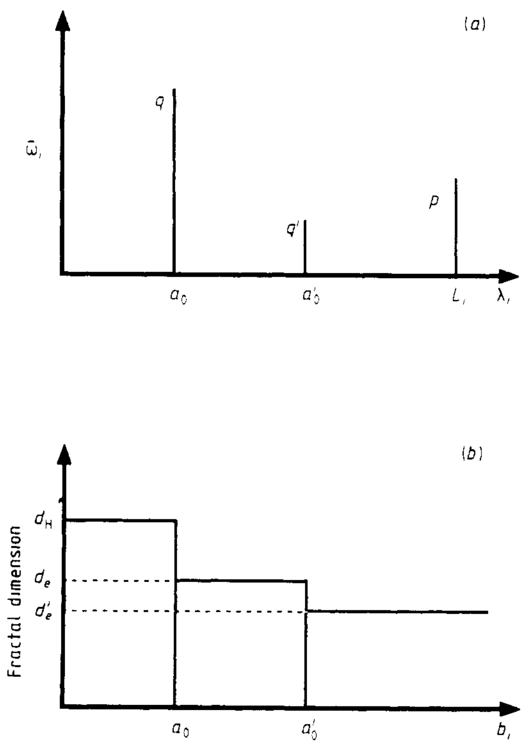

Figure 3. (a) The distribution $\bar{\omega}_i$ of sizes of the apertures λ_i at level i. (b) A schematic representation of the corresponding fractal dimensions according to expressions (11), (12), (13) (a or b) in the text.

If $a_0 < b < a_0'$ there exists an iteration step $n_0 < N$ such that $L_{n_0+1} < b < L_{n_0}$. Accordingly we have

$$p_i = \begin{cases} 1-q & i = 1, \ldots, n_0 \\ 0 & i = n_0+1, \ldots N \end{cases}$$

and ($3a$, b) gives respectively

$$M = (5-2q)^{n_0} L_{n_0} \tag{11a}$$

$$M = [1/(1-2q)]\{(1-q)[2(2-q)]^{n_0} - q3^{n_0}\} L_{n_0}. \tag{11b}$$

Using (7), we obtain for the measured fractal dimension

$$d_e = \ln(5-2q)/\ln 3 \tag{12a}$$

$$d_e = \begin{cases} \ln[2(2-q)]/\ln 3 & \text{when} \quad q < \frac{1}{2} \\ 1 & \text{when} \quad q \geq \frac{1}{2}. \end{cases} \tag{12b}$$

If $a_0' < b < a_0''$ we find in close analogy with the above

$$d_e' = \ln(5-2q-2q')/\ln 3 \tag{13a}$$

$$d_e' = \begin{cases} \ln[2(2-q-q')]/\ln 3 & \text{when} \quad q+q' < \frac{1}{2} \\ 1 & \text{when} \quad q+q' \geq \frac{1}{2} \end{cases} \tag{13b}$$

and so on for larger values of b.

Hence the experimental value of the fractal dimension depends on b (figure 3(b)) and makes a jump at each value of the aperture diameter present with a finite probability in the fractal structure.

We believe that the model presented here shows the same characteristic features as more usual random interfaces, e.g. the hull of percolation clusters. For such interfaces, we have proposed an explanation for the dependence of the measured dimension with probe size requiring two assumptions to be fulfilled: the first assumption is the asymmetry of the analysis of the interface. This is schematised in our approach in a geographical picture, where the interface is represented by a coastline. The measurement of the coast length only concerns that part of the coast which can be reached by a boat of given size. The second assumption is that the scaling laws between distributions of bay apertures and widths are different. Then the experimental value of the fractal dimension of the coast length will depend on the boat size, i.e. the experimentally observed fractal dimension of the random interface will depend on the size of probe.

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